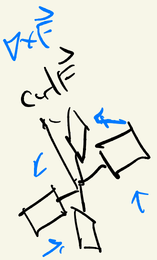
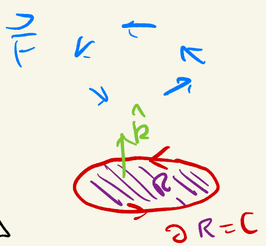
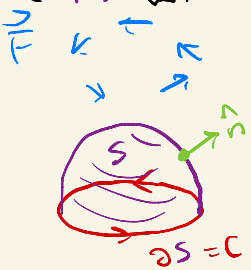
4-7:

Extending Green's theorems to 3D

- Stoke's theorem extends Green's curl-circulation theorem to <u>any</u> surface R with boundary $\partial R = C$:

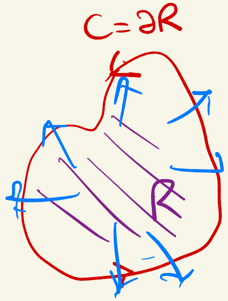


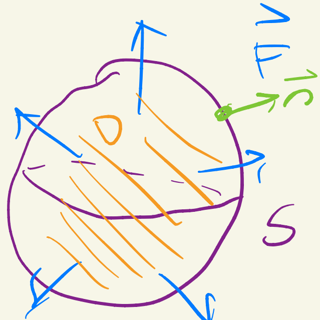




Green's (tangential)	Stokes' Theorem
$\iint_R (\nabla\times\vec{F})\cdot\hat{k}\, dA = \oint_{\partial R} \vec{F}\cdot d\vec{r}$	$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, d\sigma = \oint_{\partial S} \vec{F}\cdot d\vec{r}$

Green's Thm (normal)	Divergence Theorem
$\iint_R (\nabla\cdot\vec{F})\, dA = \oint_{\partial R} \vec{F}\cdot\vec{n}\, ds$	$\iiint_D \nabla\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot\vec{n}\, d\sigma$
Total divergence inside = outward flux boundary	Total divergence inside = outward flux boundary



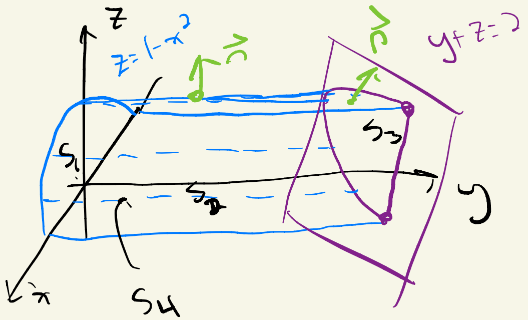


Ex:

Flux of $\vec{F} = xy\hat{i} + (y^2 + e^{xz^2})\hat{j} + \sin(xy)\hat{k}$ through S, the surface of the region D bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, $y + z = 2$

$S = \partial D$

Soln:



Flux = outward Flux

(inward flux = − outward flux)

not the only two!

2 options:

- Perform 4 surface integrals
  $S = S_1 + S_2 + S_3 + S_4$
- Divergence Thm:
  $\oiint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV$

  flux

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz}) + \frac{\partial}{\partial z}(\sin xy)$$

$$= y + 2y + 0$$

$$= 3y$$

$$\oiint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV$$

Div. Thm.

$$= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \, dy\, dz\, dx$$

$$= \frac{184}{35}$$

Note: Decompositions of the surface: $\oiint_S \vec{F}\cdot\vec{n}\, dS = \int_{S_1} \vec{F}\cdot\vec{n}\, d\sigma + \int_{S_2} '' + \int_{S_4} '' + \int_{S_3} ''$

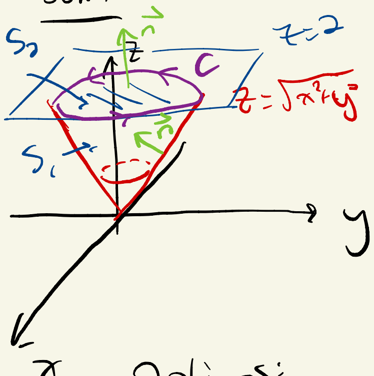
Ex:

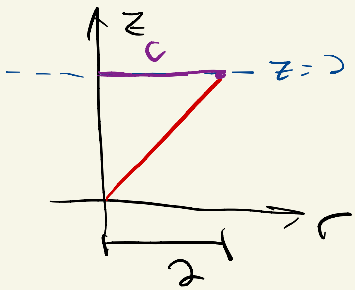
Line integral of $\vec{F} = (x^2 - y)\hat{i} + 4z\hat{j} + x^2\hat{k}$

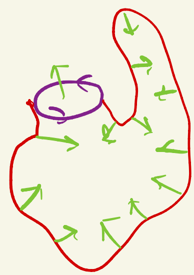
around curve C (ccw from above): intersection of $z = 2$, $z = \sqrt{x^2+y^2}$

$z = |r|$

Soln:







Options:

① Do the line integral (doesn't look too bad)

② Stokes on cone S_1

③ Stokes on lid S_2 ☆

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma$$

Stokes

curl $\vec{F}$

Notice: For lid S_2, $\vec{n} = \hat{k}$.

Also, only need $\hat{k}$ component of curl:

$\vec{G} = M\hat{i} + N\hat{j} + P\hat{k}$.

Then $\vec{G}\cdot\hat{k} = M\cdot 0 + N\cdot 0 + P\cdot 1$
$\quad\quad \hat{i} \quad\quad \hat{j} \quad\quad \hat{k}$

$= P$

only need $\hat{k}$-component of curl!

$$(\nabla\times\vec{F})\cdot\hat{k} = \left(\frac{\partial}{\partial x}(4z) - \frac{\partial}{\partial y}(x^2 - y)\right)$$

$$= 0 + 1 = 1$$

suggests easy surface integral!

3)

$$= \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{k} \, d\sigma$$

$$= \iint_{S_2} 1 \, d\sigma$$

$$= \text{Area}(S_2)$$

$$= 4\pi$$

$\blacksquare$

Ex:

$\alpha \in \mathbb{R}$

Let $\vec{F} = \langle z-y, x, y-x \rangle$ and $g(x,y,z) = \arctan(\alpha x y e^{z})$.

- Find circulation of $\vec{F}$ around the closed triangular path in xy-plane from $(0,0,0) \to (3,0,0) \to (3,3,0) \to (0,0,0)$ (straight lines)

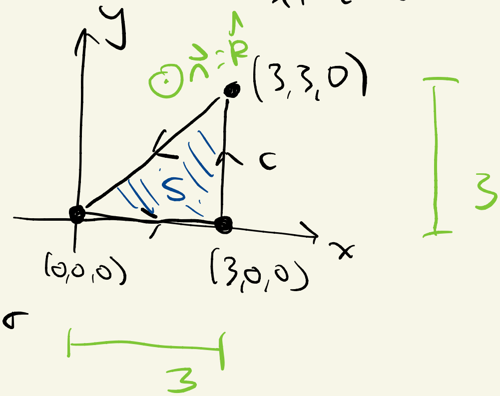
- Find circulation of $\vec{F} + \nabla g$.

vec. field + vec. field = vec. field

Soln:

$\oint_C \vec{F} \cdot d\vec{r}$ and $\oint_C (\vec{F} + \nabla g) \cdot d\vec{r}$

Set $z = 0$



① Calculate it

② Stokes'/Green's thm:

$$\oint_C \vec{F} \cdot d\vec{r} \overset{\text{stokes}}{=} \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$$

$$= \iint_S (\nabla \times \vec{F}) \cdot \hat{k}\, dA$$

in xy plane

orientation (ccw in xy)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \langle 1-0, -1-1, 1-(-1) \rangle$$

$$= \langle 1, 2, 2 \rangle$$

$\vec{F} = \langle z-y, x, y-x \rangle$

$$\Rightarrow \quad \iint_S (\nabla \times \vec{F}) \cdot \hat{k}\, dA = \iint_S \langle 1,2,2\rangle \cdot \hat{k}\, dA$$

$$= \iint_S 2\, dA$$

$$= 2 \iint_S dA$$

$$= 2\, \text{Area}(S)$$

$$= 2 \cdot \left(\frac{9}{2}\right)$$

$$= 9 \qquad \square$$

- $$\oint_C (\vec{F} + \nabla g) \cdot d\vec{r} \underset{\text{stokes}}{=} \iint_S (\nabla \times (\vec{F} + \nabla g)) \cdot \vec{n}\, d\sigma$$

Notice:

$$\nabla \times (\vec{F} + \nabla g) \overset{\text{linear}}{=} \underbrace{\nabla \times \vec{F}}_{=\langle 1,2,2\rangle} + \underbrace{\nabla \times (\nabla g)}$$

component test:
curl $(\nabla g) = 0 \Leftrightarrow \nabla g$ is conservative
gradient thm, ∇g is conservative!
$\Rightarrow \nabla \times (\nabla g) = 0$
"curl of gradient is 0"

$$= \nabla \times \vec{F}$$

$$= \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$$

$$= 9$$

Ex (cont):

Same vector field, still want circulation over the curve

$$C: (1,0,1) \to (2,0,2) \to (2,1,3) \to (1,1,2)$$

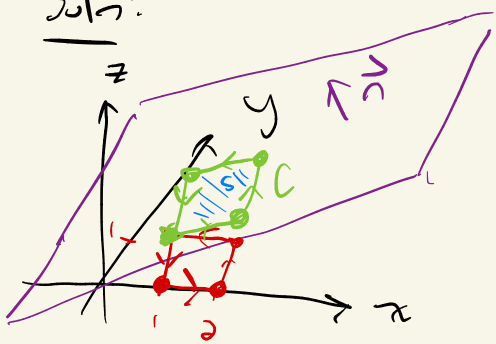
straight lines

Hint: this is the boundary curve in plane $z = x+y$

$1 \leq x \leq 2$

$0 \leq y \leq 1$

Soln:



Can calculate

$$\oint_C \vec{F}\cdot d\vec{r} \text{ directly}$$

or use Stokes

$$\oint_C \vec{F}\cdot d\vec{r} \overset{\checkmark}{=} \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma$$

would ordinarily just do explicit.

But we'll use parametrization:

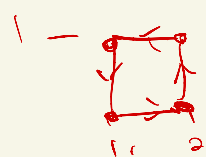
$$u = x$$

$$v = y$$

$$\Rightarrow \vec{r}(u,v) = u\hat{i} + v\hat{j} + (u+v)\hat{k}$$

$$1 \leq u \leq 2$$

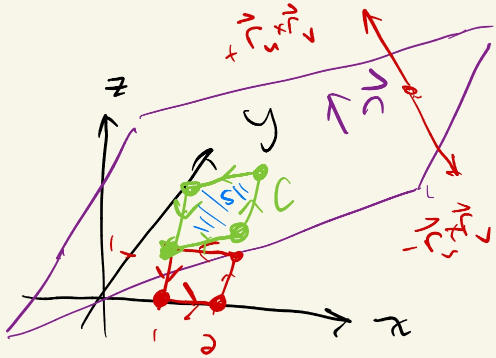
$$0 \leq v \leq 1$$



$$\vec{r}(u,v) = u\hat{i} + v\hat{j} + (u+v)\hat{k}$$

$$\Rightarrow \quad \vec{r}_u = \hat{i} + \hat{k}$$

$$\vec{r}_v = \hat{j} + \hat{k}$$



$$\Rightarrow \quad \vec{r}_u \times \vec{r}_v = -\hat{i} - \hat{j} + \hat{k}$$

Note: $\vec{r}_u \times \vec{r}_v$ is normal

to both $\vec{r}_u$ and $\vec{r}_v$

and thus the surface

$$\Rightarrow \quad \vec{n} = \underset{\text{orientation}}{\pm} \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \leftarrow \text{unit vector in direction of } \vec{r}_u \times \vec{r}_v$$

look at picture

points up!

$$+\vec{r}_u \times \vec{r}_v = -\hat{i} - \hat{j} + \hat{k}$$

$$-\vec{r}_u \times \vec{r}_v = \hat{i} + \hat{j} - \hat{k}$$

$$\Rightarrow \text{ pick } \vec{n} = + \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

$$\oint_C \vec{F} \cdot d\vec{r} \overset{\text{Stokes}}{=} \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma$$

$$\xrightarrow{\text{parametrize}} = \iint_R (\nabla \times \vec{F}) \cdot \vec{n} \, |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_R (\nabla \times \vec{F}) \cdot \vec{n} \, |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_R (\nabla \times \vec{F}) \cdot \left(\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right) |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_R (\nabla \times \vec{F}) \cdot (\vec{r}_u \times \vec{r}_v) \left(\frac{1}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \right) \cancel{|\vec{r}_u \times \vec{r}_v|} \, dA$$

$$= \iint_R (\nabla \times \vec{F}) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

$$= \int_0^1 \int_1^2 \langle 1, 2, 2 \rangle \cdot \langle -1, -1, 1 \rangle \, dA$$

$$= \int_0^1 \int_1^2 -1 \, du \, dv$$

$$= -1 \cdot \text{Area(box)}$$

$$= -1 \cdot (1) \cdot (1)$$

$$\boxed{= -1}$$

Ex:

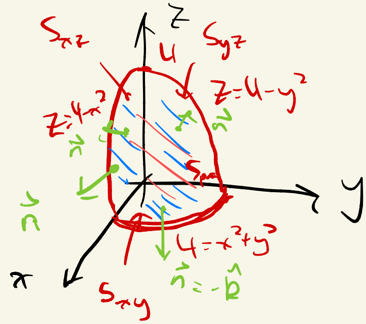
Let $\vec{F} = \langle x, y, z^2 \rangle$

a) Find an integral that will compute the outward flux of $\vec{F}$ across $S = \partial D$, where D is 3D region in positive octant under paraboloid $z = 4 - x^2 - y^2$

b) Find an integral to calculate the flux through the paraboloid component of S alone

c) Let $\vec{H} = \nabla \times \vec{G}$. Find
$$\oiint_S (\vec{F} + \vec{H}) \cdot \vec{n} \, d\sigma$$

$S = S_{para} + S_{xy} + S_{yz} + S_{xz}$



Sol'n:

① Do surface integral

② Divergence theorem

Div. Thm.

a) $\oiint_S \vec{F} \cdot \vec{n} \, d\sigma = \iiint_D \nabla \cdot \vec{F} \, dV$

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z^2$
$= 2 + 2z$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} 2 + 2z \, dz \, dy \, dx$$

Cyl.
$$= \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (2 + 2z) \, r \, dz \, dr \, d\theta$$

b) $\iint_{S_{para}} \vec{F}\cdot\vec{n}\, d\sigma$

Using divergence thm,

$$\iiint_D \nabla\cdot\vec{F}\, dV = \oiint_S \vec{F}\cdot\vec{n}\, d\sigma$$

$$\text{divergence} = \iint_{S_{para}} \vec{F}\cdot\vec{n}\, d\sigma + \underbrace{\iint_{S_{xy}} '' + \iint_{S_{yz}} '' + \iint_{S_{xz}} ''}_{\text{all of these are easy surface integrals!}}$$

- $$\iint_{S_{xy}} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_{xy}} \vec{F}\cdot(-\hat{k})\, dx\,dy$$

$$= \iint_{S_{xy}} -z^2\, dx\,dy$$

$z=0$ in xy plane $\rightarrow$ $= 0$

- $$\iint_{S_{yz}} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_{yz}} \vec{F}\cdot(-\hat{i})\, dy\,dz$$

$$= \iint_{S_{yz}} -x\, dy\,dz$$

$x=0$ in yz plane $= 0$

- $$\iint_{S_{xz}} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{S_{xz}} \vec{F}\cdot(-\hat{j})\, dx\,dz$$

$$= \iint -y\, dx\,dz \quad = 0$$

$\Rightarrow$ $\iint_{S_{para}} \vec{F}\cdot\vec{n}\, d\sigma$

$''$

$\iiint_D \nabla\cdot\vec{F}\, dV$

$''$

-1

c) $H = \nabla \times \vec{G}$

want: $\oiint_S (\vec{F} + \vec{H}) \cdot \vec{n} \, d\sigma$

well...

div. Thm

$$\oiint_S (\vec{F} + \nabla \times \vec{G}) \cdot \vec{n} \, d\sigma = \iiint_D \nabla \cdot (\vec{F} + \nabla \times \vec{G}) \, dV$$

divergence is linear

$$= \iiint_D \nabla \cdot \vec{F} + \nabla \cdot (\nabla \times \vec{G}) \, dV$$

" divergence of curl = 0 "

$\nabla \cdot (\nabla \times \vec{G}) = 0$

$$= \iiint_D \nabla \cdot \vec{F} \, dV$$

$$= -1$$